

The University of Western Australia
SCHOOL OF MATHEMATICS AND STATISTICS
BLAKERS MATHEMATICS COMPETITION

2010 Problems

The Competition begins Friday, May 14 and ends Friday, July 23.

You may use any source of information except other people. In particular, you may use computer packages (though contributed solutions are expected to have complete mathematical proofs). Extensions and generalisations of any problem are invited and are taken into account when assessing solutions.

Solutions are to be mailed or given to Greg Gamble, School of Mathematics and Statistics, The University of Western Australia, Crawley, 6009 **before 4 pm on Friday, July 23, 2010.**

Remember, you don't have to solve all the problems to win prizes!

Include a cover page with your name, address, e-mail address, University, and the number of years you have been attending any tertiary institution.

Start each problem on a new page, and write your name on every page.

1. Matrices of order 2.

Prove that for every integer $n \geq 2$ there are infinitely many $n \times n$ matrices with integer entries that are their own inverse.

2. A binary operation.

Consider a binary operation $*$ on a set S , that is, for all $a, b \in S$, $a * b$ is in S .

Prove that if for all $a, b \in S$, $(a * b) * a = b$, then for all $a, b \in S$, $a * (b * a) = b$.

3. Arithmetic equations.

Consider a system of m linear equations ($m > 1$) with real coefficients,

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i, \quad i = 1, \dots, m,$$

with a unique solution, such that

$$a_{11}, a_{12}, \dots, a_{1n}, b_1, a_{21}, a_{22}, \dots, a_{2n}, b_2, \dots, a_{m1}, a_{m2}, \dots, a_{mn}, b_m$$

is a non-trivial arithmetic progression (i.e. the common difference is not zero, so that the coefficients are not all equal).

What is that unique solution?

4. Triples.

What are all the triples (x, y, z) of positive real numbers such that $x^{y/z} = y^{z/x} = z^{x/y}$?

5. Triangle in a circle.

Let ABC be a triangle in the plane. Extend the sides AB and AC on the other side of A with segments of size $|BC|$. Similarly, extend the sides BA and BC on the other side of B with segments of size $|AC|$ and the sides CA and CB on the other side of C with segments of size $|AB|$.

Prove that the resulting six points outside the triangle are concyclic.

6. A happy class.

A number of students sit in a circle while their teacher hands out Mars bars. Each student initially has an even number of Mars bars, but not necessarily the same number. When the teacher blows a whistle, each student simultaneously gives half of his or her Mars bars to the neighbour on their right. Any student who now has an odd number of Mars bars eats one.

Show that after finitely many iterations of this procedure, all students have the same number of Mars bars.

7. Steady permutations.

A permutation π of $\{1, 2, \dots, n\}$ is *steady* if $\pi(i+1) - \pi(i) \leq 1$ for all $i \in \{1, 2, \dots, n-1\}$. How many steady permutations of $\{1, 2, \dots, n\}$ are there?

8. Pretty polynomials.

A polynomial $p(x)$ is *pretty* if, for any given point (x_0, y_0) of the plane \mathbb{R}^2 , one can find a tangent to the curve $y = p(x)$ going through (x_0, y_0) .

For which n are all polynomials of degree n pretty?

9. Alice has a birthday.

Alice: "Today is my birthday and my age is a root of this polynomial in x with integer coefficients."

Bob: "If I replace x by 7, I get 77."

Alice: "Do I look like I am 7?"

Bob: "No, indeed. I now replace x by a bigger integer N and I get 85, still not 0."

Alice: "Isn't it obvious that I am older than N ?"

How old is Alice?

10. Unavoidable numbers.

A real number $r \in (0, 1]$ is *unavoidable* if for any continuous function $f : [0, 1] \rightarrow \mathbb{R}$ such that $f(0) = f(1) = 0$, the graph of f has an horizontal chord of length r (in other words, there exists x in $[0, 1-r]$ such that $f(x) = f(x+r)$).

- Prove that all numbers of the form $1/n$ (n a positive integer) are unavoidable.
 - Prove that real numbers $r \in (0, 1)$ that are *not* of the form $1/n$ (n a positive integer) are *not* unavoidable.
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