First-order differential equations

A general first-order differential equation ($1^{\text{st}}$-order DE) may be written in the form

$$\frac{dy}{dx} = f(x, y),$$

or alternatively

$$M(x, y) \, dx + N(x, y) \, dy = 0,$$

where $f(x, y)$, $M(x, y)$ and $N(x, y)$ are arbitrary (but known!) functions of $x$ and $y$, and we wish to find a solution $y(x)$ that satisfies the DE.
There are seven main types of 1st-order differential equations, namely:

- Separable DE’s;
- Linear DE’s;
- Exact DE’s;
- Bernoulli DE’s;
- Homogeneous DE’s;
- Ricatti DE’s; and
- Abel DE’s.

General solution strategies for separable and linear DE’s were covered in MATH1001 Mathematical Methods 1 – here I will discuss solution strategies for exact, Bernoulli and homogeneous DE’s.

Unfortunately, there are no general solution strategies for Ricatti and Abel DE’s, only specific solution strategies.
Review – Separable DE’s

A 1\textsuperscript{st}-order DE is called \textit{separable} provided that the function \( f(x, y) \) may be written as the \textbf{product} of a function of \( x \) and a function of \( y \), that is \( f(x, y) = F(x)G(y) \).

Thus the variables \( x \) and \( y \) can be “separated” and placed on opposite sides of the equation; that is, given

\[
\frac{dy}{dx} = F(x)G(y),
\]

then by thinking of the derivative \( dy/dx \) as a fraction we have

\[
\frac{1}{G(y)} \, dy = F(x) \, dx,
\]

and then each side can be integrated, so that

\[
\int \frac{1}{G(y)} \, dy = \int F(x) \, dx + C,
\]
where the arbitrary integration constant $C$ includes the constants from both integrals.

We then solve this equation (if possible) for $y$, which yields the general solution of the differential equation.

If we can uniquely solve for $y$, then the solution is called the explicit solution of the differential equation, but if we cannot uniquely solve for $y$, then the solution is called the implicit solution of the differential equation.

**Example**

Consider the DE

$$\frac{dy}{dx} = \sqrt{xy}.$$
Notice that
\[
\frac{dy}{dx} = \sqrt{xy}
\]
\[
= \sqrt{x} \sqrt{y}
\]
\[
= \frac{1}{2} \frac{1}{2}
\]
\[
= x^2 y^2,
\]
so the DE is separable – separating \( x \) and \( y \) we have
\[
y^{-\frac{1}{2}} dy = x^{\frac{1}{2}} dx,
\]
and integrating both sides we have
\[
2y^{\frac{1}{2}} = \frac{2}{3} x^{\frac{3}{2}} + C,
\]
which is the implicit solution. This can be uniquely solved for \( y \), so the explicit solution is
\[
y = \left( \frac{1}{3} x^{\frac{3}{2}} + C \right)^2,
\]
where we have arbitrarily re-named the integration constant \( C \).
Review – Linear DE’s

A 1\textsuperscript{st}-order linear DE in one that may be written in the following\textit{ standard form}

\[
\frac{dy}{dx} + f(x)y = g(x),
\]

where \(f(x)\) and \(g(x)\) are arbitrary functions of \(x\) only. Note that if \(g(x) \neq 0\), the DE is \underline{not} separable.

To solve such a DE, we multiply both sides by a function \(I(x)\) such that the L.H.S. may be written

\[
I \left( \frac{dy}{dx} + fy \right) = \frac{d}{dx}(Iy),
\]

thus allowing the L.H.S. to be integrated – hence the function \(I(x)\) is called an \textit{integrating factor}.
If an integrating factor \( I(x) \) can be found, then the general solution is
\[
\frac{d}{dx}(Iy) = Ig \Rightarrow Iy = \int Ig \, dx + C \Rightarrow y(x) = \frac{1}{I(x)} \int I(x)g(x) \, dx + \frac{C}{I(x)}.
\]

How do we find the function \( I(x) \)? Since
\[
I \left( \frac{dy}{dx} + fy \right) = \frac{d}{dx}(Iy),
\]
we have by expanding the L.H.S. and using the product rule on the R.H.S, that
\[
I \frac{dy}{dx} + Ify = y \frac{dI}{dx} + I \frac{dy}{dx} \Rightarrow Ify = y \frac{dI}{dx} \Rightarrow \frac{dI}{dx} = If.
\]

This is a separable DE for \( I(x) \), with solution
\[
\frac{1}{I}dI = f \, dx \Rightarrow \ln(I) = \int f \, dx + C \Rightarrow I = \exp \left( \int f \, dx + C \right).
\]
We want the simplest possible solution for $I(x)$, so we set $C = 0$.

Hence the integrating factor $I(x) = \exp \left( \int f(x) \, dx \right)$.

**Example**

Consider the DE

$$x \frac{dy}{dx} - y - x^2 e^x.$$

In standard form we have

$$\frac{dy}{dx} - \frac{1}{x} y = x e^x,$$

so $f(x) = -\frac{1}{x}$ and $g(x) = x e^x$. Then

$$\int f(x) \, dx = \int -\frac{1}{x} \, dx = -\ln x = \ln \left( x^{-1} \right),$$
and hence
\[ I(x) = \exp \left( \int f(x) \, dx \right) = e^{\ln(x^{-1})} = x^{-1}. \]

Then
\[ \int I(x)g(x) \, dx = \int (x^{-1})(xe^x) \, dx = \int e^x \, dx = e^x, \]
and the general solution is therefore
\[ y(x) = \frac{1}{I(x)} \int I(x)g(x) \, dx + \frac{C}{I(x)} \]
\[ = \left( \frac{1}{x^{-1}} \right) (e^x) + \frac{C}{x^{-1}} \]
\[ = xe^x + Cx. \]
Exact DE’s

Consider a 1\textsuperscript{st}-order DE

\[ M(x, y) \, dx + N(x, y) \, dy = 0, \]

where \( M(x, y) \) and \( N(x, y) \) are arbitrary functions of \( x \) and \( y \).

A 1\textsuperscript{st}-order DE that can be written in this form is called an \textit{exact DE} if there exists a function \( f(x, y) \) of \( x \) and \( y \) such that

\[
\frac{\partial f}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x, y).
\]

If such a function \( f \) exists, then \( M(x, y) \, dx + N(x, y) \, dy \) may be written

\[
df = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy = 0 \quad \text{since} \quad M(x, y) \, dx + N(x, y) \, dy = 0,
\]

where \( df \) is called the \textit{total differential} of the function \( f \).
By integrating both sides the solution of $df = 0$ is simply

$$f(x, y) = C,$$

where $C$ is our constant of integration.

This equation gives us the general solution $y(x)$ implicitly, just like with some of the separable DE’s.

To check if a DE is exact, we compare $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$. If these partial derivatives are exactly equal, then the given DE is exact.

Why is this so? Recall that for a function $f(x, y)$ the first partial derivatives are

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}.$$
Since these are functions of $x$ and $y$ in their own right, they also have partial derivatives:

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x},$$

and

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}.$$

The first and last partial derivatives in the above list are the second partial derivatives with respect to $x$ and $y$ respectively, while the middle two are mixed second partial derivatives.

It turns out that the mixed partial derivatives are equal, that is

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$
This is known as Clairaut’s theorem (or equivalently Schwarz’ theorem) – this property is called “the symmetry of second derivatives” or “the equality of mixed partial derivatives”.

Now, recall that for an exact differential equation there exists a function \( f \) of \( x \) and \( y \) such that
\[
\frac{\partial f}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x, y).
\]

Differentiating the first w.r.t. \( x \) and the second w.r.t. \( y \) we have
\[
\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x},
\]
therefore since the mixed partial derivatives are equal we must have
\[
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.
\]
Example

Consider the DE

\[(3x^2y + 3y^2 - 1)\, dx + (x^3 + 6xy)\, dy = 0.\]

Here we have

\[M(x, y) = 3x^2y + 3y^2 - 1, \quad N(x, y) = x^3 + 6xy,\]

and

\[\frac{\partial M}{\partial y} = 3x^2 + 6y, \quad \frac{\partial N}{\partial x} = 3x^2 + 6y,\]

therefore \[\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\] and the DE is exact. Now let

\[\frac{\partial f}{\partial x} = M(x, y) = 3x^2y + 3y^2 - 1, \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x, y) = x^3 + 6xy.\]
So we have

\[
\frac{\partial f}{\partial x} = 3x^2y + 3y^2 - 1 \\
\frac{\partial f}{\partial y} = x^3 + 6xy
\]

\[
f = \int (3x^2y + 3y^2 - 1) \, dx \\
= x^3y + 3xy^2 - x + g(y)
\]

\[
\therefore \frac{\partial f}{\partial y} = x^3 + 6xy + g'(y)
\]

\[
\therefore g'(y) = 0 \implies g(y) = 0
\]

\[
\therefore f(x, y) = x^3y + 3xy^2 - x
\]

and therefore the implicit solution of the DE is \(x^3y + 3xy^2 - x = C\).
Alternatively, we have

\[
\frac{\partial f}{\partial x} = 3x^2y + 3y^2 - 1 \quad \quad \frac{\partial f}{\partial y} = x^3 + 6xy
\]

\[
f = \int x^3 + 6xy \, dy = x^3y + 3xy^2 + g(x)
\]

\[
\therefore \frac{\partial f}{\partial x} = 3x^2y + 3y^2 + g'(x)
\]

\[
\therefore g'(x) = -1 \Rightarrow g(x) = -x
\]

\[
\therefore f(x, y) = x^3y + 3xy^2 - x
\]

and therefore the implicit solution of the DE is again \( x^3y + 3xy^2 - x = C \).
Exact DE’s via integrating factors

Consider the DE
\[ (3xy + 2y^2 + 2) \, dx + (x^2 + 2xy) \, dy = 0. \]

Notice that the DE is not “quite” exact since
\[ M(x, y) = 3xy + 2y^2 + 2, \quad N(x, y) = x^2 + 2xy, \]
\[ \frac{\partial M}{\partial y} = 3x + 4y, \quad \frac{\partial N}{\partial x} = 2x + 2y, \]
and therefore \( \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \), but it is “almost” exact.

Sometimes 1\textsuperscript{st}-order DE’s can be turned into an exact DE by first multiplying the DE by an appropriate function \( \mu(x, y) \) of both \( x \) and \( y \), called an \textit{integrating factor} (as it enables one to integrate and hence solve the DE, just like the integrating factor for 1\textsuperscript{st}-order linear DE’s).
By multiplying the DE by \( \mu \) so we have

\[
\mu \left( 3xy + 2y^2 + 2 \right) dx + \mu \left( x^2 + 2xy \right) dy = 0,
\]

all we have to do is find a function \( \mu \) so that

\[
\frac{\partial}{\partial y} \left[ \mu \left( 3xy + 2y^2 + 2 \right) \right] = \frac{\partial}{\partial x} \left[ \mu \left( x^2 + 2xy \right) \right],
\]

and then the DE will be exact.

By using the product rule we find that

\[
\left( 3xy + 2y^2 + 2 \right) \frac{\partial \mu}{\partial y} + \mu \left( 3x + 4y \right) = \left( x^2 + 2xy \right) \frac{\partial \mu}{\partial x} + \mu \left( 2x + 2y \right).
\]

This is called a \textit{partial differential equation} (PDE) for the unknown function \( \mu(x, y) \), and they are notoriously difficult to solve. However, any solution of the PDE will serve our purpose, so we start with the simplest possible guesses and see if they lead anywhere.
If we assume that \( \mu \) is only a function of \( y \), then the PDE becomes the 1\(^{\text{st}}\)-order DE

\[
\left(3xy + 2y^2 + 2\right) \frac{d\mu}{dy} + \mu (3x + 4y) = \mu (2x + 2y) \Rightarrow \frac{d\mu}{dy} = -\frac{\mu(x + 2y)}{3xy + 2y^2 + 2},
\]

which does not make sense since the \( x \) is still present, hence \( \mu \) cannot be a function of \( y \) only.

If instead we assume that \( \mu \) is only a function of \( x \), then the PDE becomes the 1\(^{\text{st}}\)-order DE

\[
\mu (3x + 4y) = \left(x^2 + 2xy\right) \frac{d\mu}{dx} + \mu (2x + 2y) \Rightarrow \frac{d\mu}{dx} = \frac{\mu}{x},
\]

which does make sense now that the \( y \) is gone, hence we can solve for \( \mu(x) \):

\[
\frac{d\mu}{dx} = \frac{\mu}{x} \Rightarrow \int \frac{d\mu}{\mu} = \int \frac{1}{x} \Rightarrow \ln \mu = \ln x + C.
\]

Since any solution will work, we take the simplest one with \( C = 0 \), so \( \mu = x \).
Hence, returning to the original DE and multiplying through by $\mu = x$ we have

$$x \left( 3xy + 2y^2 + 2 \right) dx + x \left( x^2 + 2xy \right) dy = 0,$$

or

$$\left( 3x^2y + 2xy^2 + 2x \right) dx + \left( x^3 + 2x^2y \right) dy = 0,$$

and then

$$M(x, y) = 3x^2y + 2xy^2 + 2x, \quad N(x, y) = x^3 + 2x^2y,$$

$$\frac{\partial M}{\partial y} = 3x^2 + 4xy, \quad \frac{\partial N}{\partial x} = 3x^2 + 4xy,$$

and therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and the DE is now exact.

Then the implicit solution of the DE is

$$x^3y + x^2y^2 + x^2 = C.$$
Problems

1. Verify the implicit solution above by first integrating $\frac{\partial f}{\partial x}$, and then confirm your answer by integrating $\frac{\partial f}{\partial y}$.

2. Find the general solution of the DE

$$\left[\exp\left(\frac{y}{x}\right) - \frac{y}{x} \exp\left(\frac{y}{x}\right) - 2x\right] dx + \left[\exp\left(\frac{y}{x}\right)\right] dy = 0.$$ 

Note that this problem can only be done one way!

Solution: $y = x \ln\left(\frac{x^2 + C}{x}\right)$.

3. Find the general solution of the DE

$$y \, dx + \left(2x + \frac{e^y}{y}\right) \, dy = 0,$$

by finding an appropriate integrating factor so that the DE is exact.

Solution: $xy^2 + e^y = C$. 