Useful Theorems and Methods for First-years

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1) Pappus' theorem for surface areas and volumes of surfaces/solids of revolution

These are two very simple and somewhat intuitive equations.

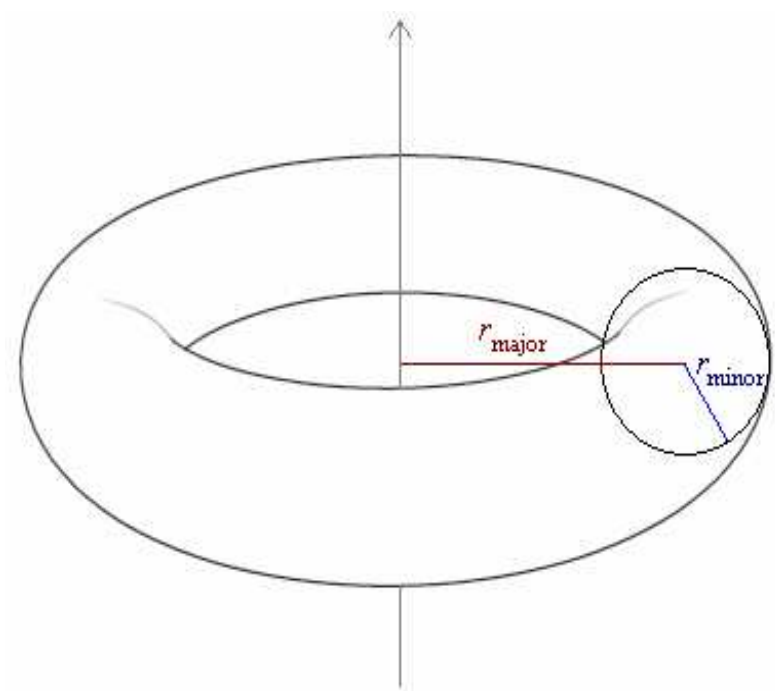
Pappus' theorem for volumes states that the volume V of the solid of revolution generated by rotating a plane region R about an axis is given by the distance traveled by the centroid of R times the area of R, provided that the axis does not pass through the interior of R, and is coplanar with R. i.e. If r is the distance from the axis to the centroid of R, then

$$V = 2\pi r A$$

where A is the area of region R.

Example: Volume of a torus.

$$V = (2\pi r_{major})(\pi r_{\min or}{}^{2})$$



Proof: Trivial, provided you know the formula for a centroid.

Pappus' theorem for surface areas

There is an analogous equation relating the centroid and surface area of surfaces of revolution:

$$A = 2\pi r S$$

where A is the surface area of the surface of revolution, and S is the arc length of the closed curve which is revolved to generate the surface. As with *Pappus' theorem* for volumes, the axis must not intersect the curve, and must be coplanar with the curve.

Example: The torus in the previous example will have surface area given by

$$A = (2\pi r_{major})(2\pi r_{\min or})$$

2) Finding the inverse of a 3x3 matrix via minors, cofactors

Typically in MATH1010, students are taught how to find the inverse of a matrix using row operations. However there is a more efficient method for finding the inverse of a 3x3 matrix:

(a) Find the determinant
(b) Find the *minors* matrix
(c) Find the *cofactors* matrix from the *minors* matrix
(d) The inverse is the transpose the cofactors matrix multiplied by the reciprocal of the determinant

The process is best illustrated with an example:

Suppose we want to invert $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$

(a) The determinant is given by

$$\begin{aligned} |A| &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \\ &= 1(1\times 4 - 3\times 2) - 3(1\times 4 - 2\times 2) + 1(1\times 3 - 2\times 1) \\ &= -1 \end{aligned}$$

This formula is easily remembered by noting that the expressions in the brackets are actually the determinants of the sub 2x2 matrix after eliminating the row and column corresponding to the coefficient of the brackets.

(b) Call the *minors* matrix M. Then m_{ij} is found by eliminating the ith row and j^{th} column in A, then finding the determinant of the remaining sub 2x2 matrix.

So $m_{11} = 1\times 4 - 3\times 2$
$m_{12} = 1\times 4 - 2\times 2$
etc...

And we eventually have $M=\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 5 & 1 & -2 \end{bmatrix}$

(c) We find the cofactors matrix by simply multiplying m_{ij} by $(-1)^{i+j}$. i.e. Pointwise multiply M by the matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Thus the *cofactors* matrix is

$$\begin{bmatrix} -2 & 0 & 1 \\ -9 & 2 & 3 \\ 5 & -1 & -2 \end{bmatrix}$$

(d) The inverse is the reciprocal of the determinant multiplied by the transpose of the *cofactors* matrix. (Note: The transpose of the cofactors matrix is called the *Adjoint* matrix.)

So $A^{-1}=\dfrac{1}{-1}\begin{bmatrix} -2 & -9 & 5 \\ 0 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}=\begin{bmatrix} 2 & 9 & -5 \\ 0 & -2 & 1 \\ -1 & -3 & 2 \end{bmatrix}$

$$A.A^{-1}=\begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 9 & -5 \\ 0 & -2 & 1 \\ -1 & -3 & 2 \end{bmatrix}=I_{3x3}$$

Important remark: When trying to invert a 3x3 matrix "by hand", the *minors-cofactors* process can often be faster and may require less calculations than using row operations. The example above only shows the 3x3 case for a general process for inverting an n by n matrix. It may be much more efficient to use the row operations method for matrices of size $n>3$.

3) Synthetic division for linear factors

Suppose you want to divide a linear factor into a polynomial of degree 1 or higher. (You may need to do this to simplify an integrand). This can be done "by hand" using long division. However a faster algorithm known as *synthetic division* might save you a bit of pen ink (and time!).

The *synthetic division* algorithm is best shown with an example:

Divide $x - 2$ into $x^3 - 5x^2 + 3x - 7$
(Note: The coefficient of x in the linear factor must be 1 before applying the algorithm.)

We first write out the coefficients of the dividend in a row, and the root of the linear factor in a box (to distinguish it from the coefficients):

1	-5	3	-7	$\boxed{2}$

Drop the first number down to the 3rd row:

1	-5	3	-7	$\boxed{2}$
1				

(a) Multiply the rightmost number in the 3rd row by the boxed number, and put the result in the 2nd row of the next column:

1	-5	3	-7	$\boxed{2}$
	2			
1				

(b) Add up the 2 numbers in that column, and put the result in 3rd row of that column:

1	-5	3	-7	$\boxed{2}$
	2			
1	-3			

Now repeat steps (a), and (b) until the 3rd row is filled:

1	-5	3	-7	$\boxed{2}$
	2	-6		
1	-3			

1	-5	3	-7	$\boxed{2}$
	2	-6		
1	-3	-3		

1	-5	3	-7	$\boxed{2}$
	2	-6	-6	
1	-3	-3		

1	-5	3	-7	$\boxed{2}$
	2	-6	-6	
1	-3	-3	-13	

The coefficients of the quotient and remainder are shown in the 3rd row. So

$$\frac{x^3 - 5x^2 + 3x - 7}{x - 2} = x^2 - 3x - 3 - \frac{13}{x - 2}$$

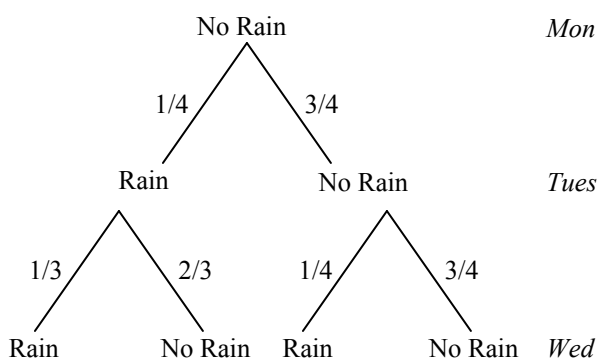
If the coefficient of the linear factor is not 1, we can re-write $ax+b$ as $a(x + b/a)$ before applying synthetic division. Also, if you are dividing by a higher order polynomial with n linear factors, then apply the algorithm n times—one for each factor, then use partial fractions.

4) Transition matrices for probability

A typical elementary conditional probability problem might be:

"If it rains on any day, there is a probability of 1/3 that it will rain again the next day. If it does not rain, there is a probability of 1/4 that it will rain the next day. If it does not rain today (say, Monday), what is the probability it will rain 3 days later (Thursday)?"

This problem can easily be solved by using a probability tree:



etc…

By considering all the paths which lead to rain on Thursday yields

P(rains on Thurs)= $\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{3}+\frac{1}{4}\cdot\frac{2}{3}\cdot\frac{1}{4}+\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{1}{3}+\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}=\frac{157}{576}$

We can arrive at the same answer by using a *probability transition matrix*:

$$\begin{array}{ccc} & & \text{Final State} \\ & & \begin{array}{cc} \text{Rain} & \text{No Rain} \end{array} \\ \text{Initial State} & \begin{array}{c} \text{Rain} \\ \text{No Rain} \end{array} & \begin{bmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{bmatrix} = P \end{array}$$

Then the probabilities of rain n days after any given day is given by P^n (the n-step transition matrix). So the probabilities for Thursday are given by P cubed:

$$\begin{array}{ccc} & & \text{Final State} \\ & & \begin{array}{cc} \text{Rain} & \text{No Rain} \end{array} \\ \begin{array}{c}\text{Initial}\\ \text{State}\end{array} & \begin{array}{c}\text{Rain}\\ \text{No Rain}\end{array} & \begin{bmatrix} 59/216 & 157/216 \\ 157/576 & 419/576 \end{bmatrix} = P^3 \end{array}$$

As before, if it did not rain today, the probability of it raining three days later is 157/576.

Why this works:
Induction provides a formal proof:
Let P be the transition matrix. Assume P^n gives the correct probabilities for time n. We find the probabilities for time $n+1$ by conditioning on the state at time n. Then it is clear the probabilities for time $n+1$ are given by $P^n P = P^{n+1}$.

The proof did not require P be 2x2. So *probability transition matrices* can be used in similar problems to this one, even if the tree is non-binary.