University of Western Australia
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1995 MATHEMATICS OLYMPIAD LECTURE NOTES
Review of logarithms

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Definition

The *logarithm* to *base* a of b, written $\log_a(b)$ is the power m that a must be raised to get b. For this definition to make sense we need $a > 0$ but $a \neq 1$ and $b > 0$. (Usually a is also an integer but this is not necessary.) More precisely then, for $1 \neq a > 0$ and $b > 0$,

$$\log_a(b) = m \iff a^m = b.$$

The $\log_a$ function should be seen as the inverse function of the $\exp_a$ function defined by:

$$\exp_a(x) := a^x, \quad x \in \mathbb{R},$$

where $a > 0$.

There are two questions here:
1. Why can we allow x to be in $\mathbb{R}$? Shouldn't x be in $\mathbb{Q}$?
2. Why do we need $a > 0$?

Suppose first that $x = p/q \in \mathbb{Q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$, and $a > 0$. Then we define

$$\exp_a(x) := (a^p)^{1/q} = \sqrt[q]{a^p},$$

where for $b = a^p > 0$, $b^{1/q} = \sqrt[q]{b}$ is the *positive real* q^{th} *root* of b and

$$a^p = a.a.\cdots.a$$

(the usual definition). We can *extend* this definition to have $x \in \mathbb{R}$ by *continuity* – essentially this means for any sequence of rational numbers $p_1/q_1, p_2/q_2, \ldots$ that approach a given $x \in \mathbb{R}$, the sequence $a^{p_1/q_1}, a^{p_2/q_2}, \ldots$ approaches some real number y – a^x is then defined to be y.

Now, why do we need $a > 0$? ... Well, if $a = 0$ then $\exp_a(x)$ is undefined for nonpositive x; and we avoid negative values of a since otherwise we encounter conflicts like the following:

- $a^{1/3}$ exists if we interpret this as the real cube root of a (then the value of $a^{1/3}$ is *negative* for negative a);
- $a^{2/6}$ ought to be interpreted as $(a^2)^{1/6}$ which is the positive 6^{th} root of the positive number a^2 (i.e. $a^{2/6}$ would necessarily be *positive* for negative a); and
- $\dfrac{1}{3} = \dfrac{2}{6}$.

So, insisting that $a > 0$ ensures that $\exp_a$ is *well-defined*.

What is a *function*? Firstly, a more general concept than a *function* is a *map* (or *mapping*). We define f to be a *map* (or *mapping*) if it is a rule that takes elements of one set, called the *domain*, to elements of another set, called the *codomain*. We say that f is a *map from* its *domain* to its *codomain*; or that f *maps* elements of its *domain* to its *codomain*. Now a *function* is a *map* with further properties. We say a map f is a *function* with domain D and codomain Y if

- f is defined for all $x \in D$; and

- for each $x \in D$, $f(x)$ is just *one* element of Y.

A particularly nice notation that emphasises this way of thinking has the following form

$$\begin{aligned} f : \text{domain} &\to \text{codomain} \\ : \quad x &\mapsto \quad f(x). \end{aligned}$$

The value $f(x)$ is called the *image* of x under f. For example, the *real square* function may be represented by

$$\begin{aligned} g : \mathbb{R} &\to \mathbb{R} \\ : x &\mapsto x^2. \end{aligned}$$

This says

g [is the function that]
maps x in $\mathbb{R}$ to x^2 in $\mathbb{R}$.

The *image* of 3 under g is 9, and more generally the *image* of x under g is x^2. You may be more familiar with defining the function g by

$$g(x) = x^2, \quad x \in \mathbb{R}.$$

This notation, however, fails to tell us what we are considering to be the *codomain*. Notice that we have not insisted that f maps at least one point of the *domain* to each point of the *codomain*. For the *real square* function g, notice that there are no points of the *domain* that are mapped to negative elements of the *codomain*. We reserve the term *range* for the subset of the *codomain* containing only the *images* of points of the *domain*. For g the *range* is $\mathbb{R}_{\geq 0}$, the set of all *nonnegative* elements of $\mathbb{R}$. (Note that *nonnegative* means *positive **or zero***).

What is an *inverse function*? Basically the *inverse function* of a function f (if one exists) should take elements of the *codomain* of f back to elements of the *domain* of f; and it should itself be a *function*. We usually write f^{-1} for the *inverse function* of the function f (when it exists). Suppose f is a function, where

$$\begin{aligned} f : D &\to \quad Y \\ : x &\mapsto f(x), \end{aligned}$$

and suppose R is the *range* of f. Then f^{-1} is defined by

$$\begin{aligned} f^{-1} : \quad Y &\to D \\ : f(x) &\mapsto x, \end{aligned}$$

so long as this definition defines a function. That is, the domain of f^{-1} is the *codomain* of f and the codomain of f^{-1} is the *domain* of f; and f^{-1} is a function if

- f^{-1} is defined for all y in its domain Y, i.e. each $y \in Y$ must be an image under f of a point in X – this can only happen if $Y = R$, in which case f is said to be *onto* (that is, if $Y = R$ we say f maps D *onto* Y); and
- for each $y \in Y = R$, $f^{-1}(y)$ is just *one* element of its codomain D, i.e. only *one* element of the domain D of f maps to any *one* element of the range R of f – if f has this property then f is said to be *one-to-one*, (this property can often be achieved by reducing the domain of f).

For the *square function* g, the codomain is $\mathbb{R}$ rather than $\mathbb{R}_{\geq 0}$ (so g is not *onto*), and if $x \neq 0$ then $-x$ and x are two points that map to the same image x^2 under g (so g is not *one-to-one*). We can make g *onto* by redefining its *codomain* to be $\mathbb{R}_{\geq 0}$ and we can make g *one-to-one* by reducing its domain to say $\mathbb{R}_{\geq 0}$. Really, this is a *new* function – so let's call it g^*, i.e.

$$\begin{aligned} g^* : \mathbb{R}_{\geq 0} &\to \mathbb{R}_{\geq 0} \\ : \quad x &\mapsto \; x^2. \end{aligned}$$

Now g^* has *inverse function* ${g^*}^{-1}$ defined by

$$\begin{aligned} {g^*}^{-1} &: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \\ &: \quad y \mapsto \sqrt{y}. \end{aligned}$$

Now let's consider the inverse function of $\exp_a$. We will need its codomain to be equal to its range, which is $\mathbb{R}_{>0}$, the set of all *positive real numbers*. So let's define, for $a > 0$

$$\begin{aligned} \exp_a &: \mathbb{R} \to \mathbb{R}_{>0} \\ &: x \mapsto \ a^x. \end{aligned}$$

This function is *one-to-one*, for all $1 \neq a > 0$. (Note that, $\exp_1(x) = 1^x = 1$ for all $x \in \mathbb{R}$, so that $\exp_1$ is not *one-to-one*.) Now finally, we can define $\log_a$ to be the inverse function of $\exp_a$, i.e. for $1 \neq a > 0$,

$$\begin{aligned} \log_a &: \mathbb{R}_{>0} \to \mathbb{R} \\ &: \quad a^x \mapsto x. \end{aligned}$$

Properties

Suppose $\boxed{1 \neq a > 0}$, $\boxed{1 \neq b > 0}$, $\boxed{x, y > 0}$ and $\boxed{m, n \in \mathbb{R}}$. Then the function $\log_a$ has the following properties.

1. $\boxed{\log_a(1) = 0}$

   since ... $a^0 = 1$ and $a \neq 0$ (in fact, we assumed $1 \neq a > 0$).

2. $\boxed{\log_a(a) = 1}$

   since ... $a^1 = a$.

3. $\boxed{\log_a(a^n) = n}$

4. $\boxed{\log_a(xy) = \log_a(x) + \log_a(y)}$

   since ...

   The statement is essentially

   $$a^m.a^n = a^{m+n}$$

   in disguise. Let $x = a^m$, $y = a^n$. Then

   $$\begin{aligned} \log_a(xy) = \log_a(a^m.a^n) &= \log_a(a^{m+n}) \\ &= m + n \\ &= \log_a(x) + \log_a(y) \end{aligned}$$

5. $\boxed{\log_a(x^n) = n \log_a(x)}$

   since ...

   The statement is essentially

   $$(a^m)^n = a^{mn}$$

in disguise. Let $x = a^m$. Then

$$\begin{aligned}\log_a(x^n) = \log_a\big((a^m)^n\big) &= \log_a(a^{mn}) \\ &= mn \\ &= nm \\ &= n\log_a(x)\end{aligned}$$

6. $\boxed{\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}}$

   since . . .

$$\begin{aligned} b^{\log_b(x)} = x &\iff (a^{\log_a(b)})^{\log_b(x)} = x \\ &\iff a^{\log_a(b).\log_b(x)} = x \\ &\iff \log_a(b).\log_b(x) = \log_a(x) \\ &\iff \log_b(x) = \frac{\log_a(x)}{\log_a(b)}. \end{aligned}$$

Additional properties

Suppose $\boxed{1 \neq a > 0}$, $\boxed{1 \neq b > 0}$ and $\boxed{x, y > 0}$. Then the function $\log_a$ has the following additional properties that are corollaries of the previous properties.

7. $\boxed{\log_a\left(\dfrac{x}{y}\right) = \log_a(x) - \log_a(y)}$

   since . . .

$$\begin{aligned}\log_a\left(\frac{x}{y}\right) &= \log_a(x.y^{-1}) \\ &= \log_a(x) + \log_a(y^{-1}), \quad \text{by Property 4.} \\ &= \log_a(x) - \log_a(y), \quad \text{by Property 5.}\end{aligned}$$

8. $\boxed{\log_a\left(\dfrac{1}{y}\right) = -\log_a(y)}$

   since . . . $\log_a\left(\dfrac{1}{y}\right) = \log_a(1) - \log_a(y) = -\log_a(y)$, by Properties 7 and 1.

9. $\boxed{\log_b(a) = \dfrac{1}{\log_a(b)}}$

   since . . . $\log_b(a) = \dfrac{\log_a(a)}{\log_a(b)} = \dfrac{1}{\log_a(b)}$, by Properties 6 and 2.

Special properties

Below we will be assuming $\boxed{a > 1}$, and $\boxed{x, y > 0}$.

10. If $a > 1$ then $\log_a$ is an increasing function.

    since ...

    Let $a > 1$. Then
    $$a^m > a^n \iff m > n.$$
    Now let $x = a^m$, $y = a^n$. Then
    $$x > y \iff \log_a(x) > \log_a(x).$$
    In particular,
    $$\log_a(x) > \log_a(x) \implies x > y,$$
    which is equivalent to saying $\log_a$ is an *increasing function*.

11. If $a > 1$ and $x > 1$ then $\log_a(x) > 0$.

    since ...

    $a > 1$ implies $\log_a$ is an increasing function. So, since $x > 1$,
    $$\log_a(x) > \log_a(1) = 0.$$