The University of Western Australia
DEPARTMENT OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2018 Problems

The Competition begins Monday, 9 July and ends Friday, 31 August 2018.

You may use any source of information except other people. In particular, you may use computer packages (though contributed solutions are expected to have complete mathematical proofs). Extensions and generalisations of any problem are invited and are taken into account when assessing solutions. Also, elegance of solution is particularly favoured and desired. We reserve the right to not read a solution if it is "messy"; we expect neat solutions (so perhaps avoid submitting a first draft).

Solutions are to be mailed or given to Greg Gamble, School of Mathematics and Statistics, The University of Western Australia, Crawley, 6009 **before 4 pm on Friday, 31 August.**

Remember, you don't have to solve all the problems to win prizes!

Instructions for solutions: Include a *cover page* with your *name*, *student ID number*, *home address*, *e-mail address*, *university* where enrolled, and the *number of years you have been attending any tertiary institution*. **Please, start each problem on a new page, number all your pages, and write your name on every page.** A hardcopy of your entry is desirable, and should be posted to the above mailing address. It is sufficient to submit a PDF-scanned copy (and recommended, even when a hardcopy has been submitted); these should be emailed to `greg.gamble@uwa.edu.au`.

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1. Circular covers

Can one completely cover a square table with sides of length 0.9 metres with two circular tablecloths whose diameters are each 1.006 metres?

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2. Cos salad

Evaluate $\left(1-\dfrac{\cos 61^\circ}{\cos 1^\circ}\right)\left(1-\dfrac{\cos 62^\circ}{\cos 2^\circ}\right)\cdots\left(1-\dfrac{\cos 119^\circ}{\cos 59^\circ}\right)$.

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3. Floating Perth

If the Earth rotated fast enough around its axis, the gravity at a point P of the globe could be counterbalanced by the vertical component of the centrifugal force at P, so that P would be in a state of "zero gravity".

To the closest minute, what would the length of a day be, if Perth were to experience "zero gravity"?

Assume the Earth is a perfect sphere of radius 6 371 km, $g = 9.81\,\mathrm{m/s^2}$, and Perth is at latitude 31.9505°S.

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4. Two spheres

In $\mathbb{R}^3$, let S be a sphere of radius 1 and let S' be a sphere of radius r passing through the centre of S.

For which value(s) of r is the area of the intersection of S' with the interior of S a maximum?

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5. Functionally composed

We define the iterations of a function f from $\mathbb{R}$ to $\mathbb{R}$ as follows:

$$f^1(x) = f(x) \quad \text{and} \quad f^n(x) = f\big(f^{n-1}(x)\big), \text{ for all integers } n > 1.$$

For instance, $f^2(x) = f\big(f(x)\big)$ and $f^3(x) = f\big(f\big(f(x)\big)\big)$.
If $f(x) = x^2 + 10x + 20$, what are all the real solutions of the equation $f^{100}(x) = 0$?

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6. Fractal covering

Take a square S. Inscribe within S a circle D, i.e. D is tangent to all four sides of S. Next we draw in each of the four corners of S, a circle tangent to D and to the two sides forming that corner. We continue in this way, infinitely many times: at each step drawing four circles (one in each corner) tangent to the circle drawn in that corner at the previous step and to the two sides forming that corner of S.
What is the ratio of the total area of all the circles we have drawn with the area of S?

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7. Prime triangle

An evil wizard has seized 3 logicians Alice, Brian and Cathy, to whom he has administered a powerful sleeping potion. On their awakening, he announces to them that he has written a prime number on each of their foreheads, and that these 3 numbers (not necessarily distinct) are the lengths of the sides of a triangle whose perimeter is also a prime number. Each logician can see the number on the foreheads of the other two, but cannot see the number on their own forehead. With a sardonic smile, the wizard announces that he will release the first logician who guesses the number on their own forehead correctly. Alice sees a 5 on Brian's forehead and a 7 on Cathy's forehead. After a long time, during which the logicians all remain silent, Alice announces that she has deduced the number written on her own forehead.
What is Alice's number and how did she reason?

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8. Random complex

We pick at random two distinct complex roots z and z' among the 2018 roots of the equation $z^{2018} = 1$.
What is the probability that $|z + z'| \geq \sqrt{2 + \sqrt{3}}$?

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9. Coincidentally

Given a tetrahedron T in $\mathbb{R}^3$, let S be the centre of the inscribed sphere, S' the centre of the circumscribed sphere, and G the centroid of T. Recall that, if A, B, C, D are the vertices of T, then the centroid G of T, is the point for which

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} + \overrightarrow{GD} = \overrightarrow{0}.$$

Is it necessarily true, that if S, S' and G coincide, then all the edges of T have the same length, so that T is a regular tetrahedron?

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10. Consecutive throws

A coin that is weighted in such a way that the probabilities of getting a tail and a head are $p = \frac{1}{3}$ and $q = \frac{2}{3}$, respectively, is thrown infinitely many times.
What is the probability that the first occurrence of 4 consecutive heads is before the first occurrence of 3 consecutive tails?